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Biframe Bundle Geometry: An Extension of Rainich–Misner–Wheeler Theory to Include Sources

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Recently the original theory of Rainich, Misner, and Wheeler (RMW) has been shown to have a natural reformulation in terms of a new principal fiber bundle, namely the bundle of biframes L^2M over spacetime. We extend this new formalism further and show that the original RMW program can be generalized to include Einstein-Maxwell spacetimes with geometrical sources. The assumptions of a Riemannian connection one-form on the linear frame bundle LM and a general connection one-form on L^2M necessarily imply the existence of a difference form K. A generalization of the standard RMW theorem is developed which provides the necessary and sufficient conditions on an arbitrary triple (M, g, K) in order for this triple to be an Einstein-Maxwell spacetime with geometrical sources. All sources for the field equations associated with such spacetimes are geometrical, as they are constructible from the metric g, the difference form K, and their derivatives. The extension of the RMW program presented here introduces a second complexion vector, in addition to the standard RMW complexion vector, and the formalism reduces, in the special case of no sources, to the standard RMW program.

1. INTRODUCTION

The already unified theory of Rainich (1925) and Misner and Wheeler (1957) (RMW) provides a geometrization of source-free Einstein-Maxwell spacetimes within the standard arena of four-dimensional Riemannian geometry. Indeed, the usual RMW program consists of providing necessary and sufficient conditions on an arbitrary four-dimensional Riemannian geometry (M, g) in order for this geometry to be a source-free Einstein-Maxwell spacetime. However, within the standard arena of Riemannian geometry there has never been a method to generalize the RMW program to include Einstein-Maxwell spacetimes with sources.

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The purpose of this paper is to show that the original RMW theory can be extended to include geometrical sources by generalizing the geometrical arena from the linear frame bundle LM to a new principal fiber bundle over spacetime, namely the bundle of biframes L^2M. The differential geometry associated with L^2M has recently been developed by Hammon and Norris (1990*b*). A key geometrical development which allows this generalization is the introduction of a difference form K on L^2M. We show that if the spacetime is space and time orientable, then K exists globally; otherwise K always exists locally, and we denote this structure by a triple (M, g, K). A generalization of the standard RMW theorem is developed which provides the necessary and sufficient conditions on an arbitrary triple (M, g, K) in order for this triple to be an Einstein-Maxwell spacetime with geometrical sources. All sources for the field equations associated with such spacetimes are geometrical, as they are constructible from the metric g, the difference form K, and their derivatives. This generalization of the RMW program provides a completely geometrical extension of the formalism developed previously by Hammon and Norris (1990*a*,*c*).

As described above, we wish to extend the original RMW program to include sources. An Einstein-Maxwell spacetime with sources is any four-dimensional Riemannian spacetime (M, g) which satisfies

$$\tilde{G}_{\mu\nu} = (f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \nu}) + T^{s}_{\mu\nu} \tag{1.1}$$

$$\nabla_{\mu} f^{*\lambda\mu} = J^{\lambda}_{m} \tag{1.2}$$

$$\nabla_{\mu} f^{\lambda\mu} = J^{\lambda}_{e} \tag{1.3}$$

where $\tilde{G}_{\mu\nu}$ is the Riemannian Einstein tensor, $f^{\lambda\mu}$ is the Maxwell field strength, and $f^{*\lambda\mu} = \frac{1}{2}\mathscr{E}^{\lambda\mu\alpha\beta} f_{\alpha\beta}$ is the Hodge dual of $f_{\alpha\beta}$. The tensor T^s represents all nonelectromagnetic sources for the Einstein tensor, while J_m and J_e represent magnetic and electric currents, respectively, for the Maxwell equations. Note that within this standard formalism the entities T^s, J_m, J_e, and $f^{\lambda\mu}$ are all nongeometrical, since they cannot be constructed, in general, from the metric g and its derivatives. In the above relations ∇_μ denotes the local covariant derivative operator associated with the given Riemannian connection. This convention follows Schouten (1953) and we will use his notation throughout. Furthermore, we will mainly follow Kobayashi and Nomizu (1963) when considering the differential geometry associated with fiber bundles.

The original work of Rainich (1925) on the algebraic structure of the electromagnetic field was the basis of the already unified theory of nonnull, source-free Einstein-Maxwell spacetimes as presented by Misner and Wheeler (1957). Such spacetimes correspond to relations (1.1)-(1.3) with the restrictions $T^s = 0$, $J_m = 0$, and $J_e = 0$, as well as the nonnull electromagnetic condition $f^*_{\theta\phi} f^{\theta\phi} \neq 0$. The original RMW theory was shown to include null electromagnetic fields ($f^*_{\theta\nu} f^{\theta\phi} = 0$) as well by Geroch (1966). Since the original application to electromagnetic fields, the "Rainich program" of geometrization has been developed for other fields as well. These cases amount to relations (1.1)–(1.3) with $f_{\mu\nu} = f^*_{\mu\nu} = 0$, that is, to special forms of T^s. In particular, Kuchar (1963) extended the Rainich program within the context of Riemannian geometry to both real and charged scalar fields. Moreover, by generalizing the geometrical arena to allow for nonzero torsion on the linear frame bundle, Kuchar (1965) was able to geometrize fermion fields as well. Recently, again within the framework of Riemannian geometry, Coll and Ferrando (1989) extended the earlier works of Taub (1963) and McVittie (1956) to provide a geometrization of a thermodynamic perfect fluid.

All of the above programs deal with special cases of the coupled Einstein–Maxwell equations with sources, as given in (1.1)–(1.3); however, until recently no one has considered a geometrization of these coupled equations in their full generality. Hammon and Norris (1990*a*) showed that the original RMW program has a natural reformulation in terms of the geometry of a new principal fiber bundle, namely the bundle of biframes L^2M. Furthermore, it was shown that the original RMW program could be extended to include "Einstein-Maxwell spacetimes with partially geometrical sources." The field equations associated with such spacetimes have the same form as (1.1)–(1.3) with the following exception: the electric and magnetic currents J_m and J_e are geometrized in a natural manner in terms of the geometry associated with L^2M.

The above program was "partially geometrical" in that it was necessary to assume a given nongeometrical, nonelectromagnetic source-stress tensor T^s as in (1.1), and thus a complete geometrization of the Einstein equation was not possible. In this paper we extend the previous formalism in a completely geometrical manner in terms of the geometry associated with the linear frame bundle LM and the biframe bundle L^2M. Indeed, we will show using standard decomposition theorems that the assumptions of a Riemannian connection one-form on LM and a general connection one-form on L^2M necessarily imply the existence of a new geometrical difference one-form K. This geometrical object will lead to a geometrization of the Einstein equation, as in (1.1), including a geometrization of the previously nongeometrical T^s.

The fundamental idea which allows the above generalization of the standard RMW program is the introduction of a new geometrical arena, namely the principal bundle of biframes L^2M over spacetime. The geometry associated with the biframe bundle has recently been developed by Hammon and Norris (1990*b*), and a review of the main features of this fiber bundle appear in Section 2 of this paper. As a means to better understand the new geometry associated with this bundle, as well as the relations between the geometries of LM and L^2M, we present the following intuitive discussion.

The geometry associated with a four-dimensional Riemannian spacetime is usually developed by considering vectors as the fundamental geometrical objects on the spacetime. Since the set of all vectors at a spacetime point forms a four-dimensional vector space, a basis (linear frame) of this space thus consists of four independent vectors. The linear frames transform among themselves by elements of the general linear group $Gl(4)$, and the corresponding geometrical arena is the bundle of linear frames LM over spacetime. The linear frame bundle LM is thus the natural arena in which to study the geometry based fundamentally on vectors on spacetime.

However, from a geometrical point of view, one could equally well consider *bivectors*, or antisymmetric contravariant tensors of rank two, as the fundamental geometrical building blocks on spacetime rather than vectors. The set of all bivectors at a point of the four-dimensional spacetime forms a six-dimensional vector space, and a basis of this space thus consists of six independent bivectors at each point of the spacetime. Such a basis will be called a *biframe*, and the infinite number of biframes at each point transform among themselves by elements of $Gl(6)$. A new principal fiber bundle, which Norris (1980) called the bundle of biframes L^2M, can now be constructed over spacetime.

The geometry of the bundle of biframes L^2M is analogous to, but distinct from, that on the linear frame bundle LM. The biframe bundle supports a soldering form which is a two-form, as compared to the soldering one-form on LM. Thus, the corresponding torsion of a given connection one-form on L^2M, or bitorsion, is a three-form. We emphasize that the bitorsion three-form on L^2M is, in general, distinct from the linear torsion two-form associated with a given connection one-form on LM. The bitorsion structure equation and the associated Bianchi identity will play a central role in the geometrization of the Maxwell equations which occurs later in this paper.

As described above, the geometry associated with the frame bundle LM is usually distinct from that associated with the biframe bundle L^2M; however, the vector geometry associated with LM can induce some, but not all, of the bivector geometry associated with L^2M. Indeed, a given connection one-form $\tilde{\omega}$ on LM can induce a connection one-form $\mathring{\omega}$ on L^2M, and the associated curvature two-form $\tilde{\Omega}$ on LM can induce a curvature two-form $\mathring{\Omega}$ on L^2M. This concept of induced geometry will play a central role in the later formalism and we consider this notion in detail in Section 3.

An intuitive example of biframe bundle geometry which is induced from geometry on the frame bundle, as described above, is the following. Assume a Riemannian connection one-form on LM and let $s\colon U \to LM$, $U \subseteq M$, be a local section of LM. The components of the pullback of the Ricci identities for a bivector Λ from LM to M in the local section s of LM can be written as

$$\nabla_{[\mu}\nabla_{\nu]}\Lambda_{\gamma\delta} = \tfrac{1}{2}(-\tilde{R}^{\sigma}_{\mu\nu\gamma}\Lambda_{\sigma\delta} - \tilde{R}^{\sigma}_{\mu\nu\delta}\Lambda_{\gamma\sigma}) \tag{1.4}$$

where $\tilde{R} = s^*\tilde{\Omega}$ is the Riemannian curvature tensor on M.

These identities can be rewritten in the form

$$\nabla_{[\mu}\nabla_{\nu]}\Lambda_{\gamma\delta} = \tfrac{1}{2}\mathring{R}^{\theta\phi}_{\mu\nu\gamma\delta}\Lambda_{\theta\phi} \tag{1.5}$$

where we have defined

$$\mathring{R}^{\theta\phi}_{\mu\nu\gamma\delta} = \tfrac{1}{2}(\tilde{R}^{\theta}_{\mu\nu\gamma}\delta^{\phi}_{\delta} - \tilde{R}^{\phi}_{\mu\nu\gamma}\delta^{\theta}_{\delta} - \tilde{R}^{\theta}_{\mu\nu\delta}\delta^{\phi}_{\gamma} + \tilde{R}^{\phi}_{\mu\nu\delta}\delta^{\theta}_{\gamma}) \tag{1.6}$$

The curvature tensor $\mathring{R}$ given in (1.6) is a local biframe curvature tensor which is induced by the Riemannian curvature tensor $\tilde{R}$. Indeed, the Riemannian connection one-form $\tilde{\omega}$ on LM induces a connection one-form $\mathring{\omega}$ on L^2M, and the corresponding curvature two-form $\tilde{\Omega}$ on LM induces a curvature $\mathring{\Omega}$ on L^2M, where $\mathring{R} = \sigma^*\mathring{\Omega}$ is the pullback of the curvature two-form $\mathring{\Omega}$ from L^2M to M in the local section σ of L^2M.

Note that a general connection one-form ω on L^2M will not be, in general, just the induced connection one-form $\mathring{\omega}$ on L^2M, and the corresponding curvature two-form Ω on L^2M will therefore not be the same as the induced curvature $\mathring{\Omega}$ whose local components appear in (1.6). Given a Riemannian connection one-form $\tilde{\omega}$ on LM and a general connection one-form ω on L^2M, we define the difference one-form K on L^2M by $K = \omega - \mathring{\omega}$, where $\mathring{\omega}$ is the connection one-form on L^2M which is induced by $\tilde{\omega}$. Furthermore, we denote any four-dimensional spacetime manifold M, which has the geometrical structure on LM and L^2M as described above, by a triple (M, g, K), where g is the metric tensor. Further details concerning the difference form K are developed in Section 4.

In this paper we first consider geometrical developments in Sections 2–4 and follow with physical considerations in Sections 5–9. In particular, a sketch of the main geometrical features associated with the bundle of biframes is given in Section 2. These geometrical developments are continued in Sections 3 and 4 as the notions of induced geometry and the natural difference form K are considered in detail. In Section 5 we present a review of the standard RMW theory. The concept of algebraic RMW spacetimes is generalized to the bundle of biframes in Section 6 and a natural geometrization of the Einstein equation with sources is also provided. In Section 7 it is demonstrated that the biframe bundle is a natural arena in which to formulate the generalized RMW program. The Maxwell equations with sources are geometrized via the bitorsion structure equation on L^2M in Section 8. Furthermore, an important theorem related to the RMW differential condition also appears in this section. A generalization of the standard RMW theorem to include geometrical sources is presented in Section 9, and it is shown that the new formalism reduces, in the special case of no sources, to the standard RMW theorem. Finally, in Section 10 we present conclusions and discuss implications for future work.

2. THE BUNDLE OF BIFRAMES

Before considering a sketch of the structure of the biframe bundle [a more detailed account can be found in Hammon and Norris (1990*b*)], a sketch of the standard frame bundle LM will be given. Assume a four-dimensional spacetime manifold M. The frame bundle LM is a principal fiber bundle with structure group $Gl(4)$. A point $u \in LM$ can be written as $u=(p, e_\alpha)$, where (e_α) is a basis of T^1M_p, with dual basis $(\mathscr{E}^\alpha)$ $[\mathscr{E}^\alpha(e_\beta)=\delta^\alpha_\beta, \alpha, \beta=1,\ldots,4]$. The projection π: $LM \to M$ is defined by $\pi(u)=p$. A local section (tetrad field) s: $U \to LM$, $U \subseteq M$, can be defined as

$$s(p)=(p, e_\alpha|_p)$$

The frame bundle LM is unique among $Gl(4)$ principal bundles over spacetime in that it supports an object called the soldering form θ[see, for example, Trautman (1970) and Norris *et al.* (1980)]. The soldering form on LM is an $\mathbb{R}^4$-valued one-form, that is, θ: $T_uLM \to \mathbb{R}^4$ and is defined by $\theta_u(X)=\mathscr{E}^\alpha[d\pi(X)]r_\alpha$, where (r_α) is the standard basis of $\mathbb{R}^4$, $u=(p, e_\alpha)$, and $X \in T_uLM$. The soldering form on LM is characterized by the following properties:

(a) θ is an $\mathbb{R}^4$-valued one-form on LM.
(b) $R^*_g\theta = g^{-1}\cdot\theta$, $\forall g \in Gl(4)$.
(c) $\theta(X)=0$ if and only if $d\pi(X)=0$.

In (b) the "dot" denotes the standard action of $Gl(4)$ on $\mathbb{R}^4$.

Given a connection one-form $\tilde{\omega}$ on LM, the curvature and torsion two-forms are defined by (Kobayashi and Nomizu, 1963)

$$\tilde{\Omega}=\tilde{D}\tilde{\omega}=d\tilde{\omega}+\tilde{\omega}\wedge\tilde{\omega} \tag{2.1}$$

$$\tilde{\Theta}=\tilde{D}\theta=d\theta+\tilde{\omega}\wedge\theta \tag{2.2}$$

where $\tilde{D}$ denotes the exterior covariant derivative with respect to $\tilde{\omega}$. The torsion $\tilde{\Theta}$ is an $\mathbb{R}^4$-valued two-form. Pulled back in a local gauge $s(p)=(p, e^i_\alpha\partial_i|_p)$ of LM, (2.2) takes the form

$$_s\tilde{\Theta}^\alpha_{jk}=2(\partial_{[j}\mathscr{E}^\alpha_{k]}+\Gamma^\alpha_{[j|\beta|}\mathscr{E}^\beta_{k]}) \tag{2.3}$$

The biframe bundle L^2M is defined over the same four-dimensional spacetime manifold M. The associated tensor spaces are the spaces AT^2M_p of antisymmetric rank-two tensors at $p \in M$. As a vector space, AT^2M_p is six-dimensional. Let (t_a) $(a=1,\ldots,6)$ be a basis (a *biframe*) of AT^2M_p. The dual basis is denoted by (τ^a) and satisfies $\tau^a(t_b)=\delta^a_b$. Since the t_a are antisymmetric rank-two tensors (*bivectors*), they can be written as

$$t_a = \tfrac{1}{2} t_a^{\alpha\beta}(e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha) \overset{d}{=} t_a^{\alpha\beta}(e_\alpha \wedge e_\beta), \qquad \alpha < \beta$$

for (e_α) a basis of T^1M. Further, the t_a transform, in general, under $Gl(6)$. That is, if $g \in Gl(6)$, $g=(g^a_b)$ $(a, b=1,\ldots,6)$, then a new biframe basis $\bar{t}_a$ can be defined by $\bar{t}_a = t_b g^b_a$. Since there are six independent bivectors in the biframe, one needs the full $Gl(6)$ transformation group.

The biframe bundle L^2M is a principal fiber bundle with structure group $Gl(6)$. A point $u \in L^2M$ can be written as $u=(p, t_a)$, where $(t_a)_p$ is a biframe at $p \in M$. The projection π: $L^2M \to M$ is defined by $\pi(u)=p$. A local section σ: $U \to L^2M$, $U \subseteq M$, is defined by $\sigma(p)=(p, t_a|_p)$ for all points $p \in U$. The right action of $Gl(6)$ on L^2M is defined by R_g: $L^2M \to L^2M$ such that $R_g u = u \cdot g = (p, t_a g^a_b)$, $\forall g \in Gl(6)$. A connection one-form on L^2M is a $gl(6)$-valued one-form with the standard properties of a connection.

The biframe bundle does support a generalized soldering form. However, the striking difference is that this soldering form is a *two-form*, as opposed to the one-form on LM. Let β be an $\mathbb{R}^6$-valued two-form on L^2M, that is, β_u: $T_uL^2M \times T_uL^2M \to \mathbb{R}^6$, defined by

$$\beta_u(X, Y) = \tau^a(d\pi(X), d\pi(Y))r_a$$

for $X, Y \in T_uL^2M$, $u=(p, t_a)$, and (τ^a) dual to (t_a). Here and in the following (r_a) denotes the standard basis of $\mathbb{R}^6$.

The soldering form on L^2M has the following properties:

(a) β is an $\mathbb{R}^6$-valued two-form on L^2M.
(b) $R_g^*\beta = g^{-1}\cdot\beta$, $\forall g \in Gl(6)$.
(c) $\beta(X, Y)=0$ if $d\pi(X)=0$ and/or $d\pi(Y)=0$.

In (b) the "dot" denotes the standard action of $Gl(6)$ on $\mathbb{R}^6$.

Given a connection one-form ω on L^2M, the curvature of ω is the $gl(6)$-valued two-form defined in the standard way by

$$\Omega = D\omega = d\omega + \omega \wedge \omega \tag{2.4}$$

However, the bitorsion of the connection is an $\mathbb{R}^6$-valued tensorial *three-form*, defined by

$$\Theta = D\beta = d\beta + \omega \wedge \beta \tag{2.5}$$

The bitorsion and the curvature on L^2M satisfy the first and second Bianchi identities

$$D\Theta = \Omega \wedge \beta \tag{2.6}$$

$$D\Omega = 0 \tag{2.7}$$

Let σ: $U \to L^2M$, $U \subseteq M$, be a local section of L^2M with $\sigma(p) = (p, t_a|_p)$, $\forall p \in U$. The components of the pullback of the curvature take the standard form (where $\Gamma = \sigma^*\omega$, $R = \frac{1}{2}\sigma^*\Omega$)

$$R^{b}_{\alpha\beta a} = 2(\partial_{[\alpha}\Gamma^{b}_{\beta]a} + \Gamma^{b}_{[\alpha|c|}\Gamma^{c}_{\beta]a}) \tag{2.8}$$

Associated objects can be defined by using a biframe (t_a) and its dual (τ^a) to express Lie algebra indices as spacetime indices. For example, $R^{b}_{\alpha\beta a}$ can be reexpressed as $R^{\sigma\rho}_{\alpha\beta\gamma\delta} = R^{b}_{\alpha\beta a}\tau^{a}_{\gamma\delta}t^{\sigma\rho}_{b}$. The local components of the pullback of the bitorsion (2.5) and the first Bianchi identity (the bitorsion Bianchi identity) (2.6) take the forms

$${}_{\sigma}\Theta^{a}_{\alpha\beta\gamma} = \partial_{[\alpha}\tau^{a}_{\beta\gamma]} + \Gamma^{a}_{[\alpha|b|}\tau^{b}_{\beta\gamma]} \tag{2.9}$$

$$\partial_{[\alpha}({}_{\sigma}\Theta)^{a}_{\beta\gamma\delta]} + \Gamma^{a}_{[\alpha|b|}({}_{\sigma}\Theta)^{b}_{\beta\gamma\delta]} = \tfrac{1}{2}R^{a}_{[\alpha\beta|b|}\tau^{b}_{\gamma\delta]} \tag{2.10}$$

respectively, where ${}_{\sigma}\Theta = \sigma^*\Theta$. Here and in the following we use a left subscript to denote the gauge when needed for clarity. Since the bitorsion is a three-form, we define an equivalent one-form by

$${}^{*}_{\sigma}\Theta^{\lambda a} \stackrel{d}{=} \tfrac{1}{2}\mathscr{E}^{\lambda\alpha\beta\gamma}({}_{\sigma}\Theta)_{\alpha\beta\gamma}$$

where we use the unconventional factor of $\frac{1}{2}$ for later convenience.

3. GEOMETRY ON L^2M INDUCED BY GEOMETRY ON LM

In general, geometries based on connections on LM and L^2M are completely independent. However, we will show below that the geometry of the frame bundle can influence that of the biframe bundle in certain special cases. In particular, we will describe below the special geometry on L^2M which is induced by geometry on LM. A relationship is then developed in the following section between this special geometry on L^2M and geometry on L^2M which is not induced in this manner. This relationship ultimately leads to a decomposition of the Ricci tensor associated with L^2M into two tensors, one of which is the Einstein tensor associated with a Riemannian connection on LM. *Throughout the remainder of this paper we will assume a Riemannian connection one-form $\tilde{\omega}$ on LM and a general connection one-form ω on L^2M.*

A given connection $\tilde{\omega}$ on LM can induce a connection $\mathring{\omega}$ on L^2M in the following manner. The covariant derivative operator obeys the following standard relations when it acts on a basis $(e_\beta)_p$ with dual basis $(\mathscr{E}^\alpha)_p$, namely $\nabla_\alpha e_\beta = \Gamma^\gamma_{\alpha\beta} e_\gamma$, where $\Gamma^\beta_{\mu\alpha}$ are the components of the Levi-Civita connection in the basis (e_α). Furthermore, it also obeys the standard rules for tensor products. Thus, acting on a bivector basis, we obtain

$$\begin{aligned}\nabla_\alpha(e_\beta \wedge e_\gamma) &= (\delta^\theta_\gamma \Gamma^\sigma_{\alpha\beta} - \delta^\theta_\beta \Gamma^\sigma_{\alpha\gamma})(e_\sigma \wedge e_\theta) \\ &\stackrel{d}{=} \mathring{\Gamma}^{\theta\sigma}_{\alpha\beta\gamma}(e_\theta \wedge e_\sigma)\end{aligned} \tag{3.1}$$

The objects $\mathring{\Gamma}^{\theta\sigma}_{\alpha\beta\gamma}$ are the components of a biframe connection induced by the given Riemannian connection on LM. The associated connection

$$\mathring{\Gamma}^a_{\alpha b} = t^{\beta\gamma}_b \mathring{\Gamma}^{\theta\sigma}_{\alpha\beta\gamma} \tau^a_{\theta\sigma} - t^{\mu\nu}_b \partial_\alpha \tau^a_{\mu\nu}$$

satisfies the Kozul type relation $\nabla_\alpha t_a = \mathring{\Gamma}^b_{\alpha a} t_b$.

Thus, the local components of a biframe connection $\mathring{\omega}$ which are induced by those of a connection one-form $\tilde{\omega}$ on LM can be defined by

$$\mathring{\Gamma}^b_{\mu a} \stackrel{d}{=} -t^{\alpha\beta}_a \nabla_\mu \tau^b_{\alpha\beta} \tag{3.2}$$

The connection will be called the *biframe connection induced by the connection* $\tilde{\omega}$ *on LM.*

We note that the existence of such a connection $\mathring{\omega}$ which is globally defined on L^2M depends on an imbedding of $LM(M, Gl(4))$ into $L^2M(M, Gl(6))$ as principal fiber bundles [see, for example, Kobayashi and Nomizu (1963)]. The author considered this problem (Hammon, 1989) and found special cases in which such an imbedding can be accomplished. In particular, if the spacetime (M, g) is space and time orientable [see, for example, Bleecker (1981)], it is possible to imbed the corresponding orthonormal frame bundle $\overline{OM}(M, L^\uparrow_+)$ into the biframe bundle $L^2M(M, Gl(6))$, where $L^\uparrow_+$ is the proper Lorentz subgroup of $O(3, 1)$. A second case involves a new principal fiber bundle $\overline{LM}(M, \overline{Gl}(4))$, which is related to $LM(M, Gl(4))$. This new bundle identifies points (p, e_α) and $(p, -e_\alpha)$ of LM and locally cannot be distinguished from LM. However, the new bundle $\overline{LM}(M, \overline{Gl}(4))$ can quite generally be imbedded in $L^2M(M, Gl(6))$. Further development of the bundle $\overline{LM}(M, \overline{Gl}(4))$ remains a prospect for future investigation. However, the general case of imbedding $LM(M, Gl(4))$ into $L^2M(M, Gl(6))$ does not appear possible. Thus, when such a connection $\mathring{\omega}$ does not exist globally, one may still define such a connection locally as in (3.2).

The induced biframe connection $\mathring{\omega}$ leads to a corresponding induced biframe curvature whose local components we will denote as $\mathring{R}^b_{\mu\nu a}$. Of particular interest in later applications is the associated object

$$\mathring{R}^{\gamma\delta}_{\mu\nu\alpha\beta} = \mathring{R}^b_{\mu\nu a} \tau^a_{\alpha\beta} t^{\gamma\delta}_b$$

which is constructible from the $\mathring{\Gamma}^{\gamma\delta}_{\mu\alpha\beta}$. The biframe curvature $\mathring{R}$ which is induced from the curvature $\tilde{\Omega}$ on LM was discussed in the introduction and takes the explicit form given in relation (1.6). As with any biframe curvature, the object $\mathring{R}$ has the index symmetries

$$\mathring{R}^{\theta\phi}_{\mu\nu\gamma\delta} = \mathring{R}^{[\theta\phi]}_{[\mu\nu][\gamma\delta]}$$

Note that if $g_{\alpha\beta}$ are the local components of the given metric g on M, a bimetric $\mathring{g}$ induced by g can be defined by $\mathring{g}^{\alpha\beta\gamma\delta} = \frac{1}{2}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma})$. A number of interesting relations and contractions of the induced curvature $\mathring{R}$ can be derived. In particular, we form the "Ricci tensor" of the induced biframe curvature, namely

$$\begin{aligned}\mathring{R}^{\alpha\phi} &\stackrel{d}{=} \mathring{g}^{\alpha\nu\gamma\delta}\mathring{R}^{\mu\phi}_{\mu\nu\gamma\delta} \\ &= \tilde{R}^{\alpha\phi} - \tfrac{1}{2}g^{\alpha\phi}\tilde{R} = \tilde{G}^{\alpha\phi} \qquad (3.3)\end{aligned}$$

This particular contraction gives the entire Einstein tensor of the Riemannian spacetime. This relation between the induced biframe curvature and the Einstein tensor provides an intimate link between the biframe bundle geometry and the standard Riemannian geometry associated with the general theory of relativity.

4. A FUNDAMENTAL DIFFERENCE FORM

Some of the geometry on the biframe bundle can be induced from that on the frame bundle. Indeed, we have shown in the previous section that a given Riemannian connection one-form $\tilde{\omega}$ on LM can induce a connection one-form $\mathring{\omega}$ on L^2M. It is clear, however, that a general connection one-form ω on L^2M need not be induced in this manner. A precise relation between a general connection one-form ω on L^2M and an induced connection one-form $\mathring{\omega}$ on L^2M is developed below.

Given a general connection one-form ω on L^2M and a connection one-form $\mathring{\omega}$ on L^2M which is induced by a given Riemmanian connection one-form $\tilde{\omega}$ on LM, as described above, we define a *difference form* $K = \omega - \mathring{\omega}$. Following Kobayashi and Nomizu (1963), we note that K is a tensorial one-form on L^2M of type (ad, $gl(6)$) and it is not difficult to show that K is uniquely related to a type (2, 3) tensor field on M. The reader should consult Hammon and Norris (1990*b*) for further details.

The local components of the associated difference tensor K now take the form

$$K^b_{\mu a} \stackrel{d}{=} \Gamma^b_{\mu a} - \mathring{\Gamma}^b_{\mu a} \qquad (4.1)$$

Objects associated to $K^b_{\mu a}$ will be denoted in the standard manner as $K^b_{\mu\alpha\beta} = K^b_{\mu a}\tau^a_{\alpha\beta}$ and $K^{\gamma\delta}_{\mu\alpha\beta} = K^b_{\mu a}\tau^a_{\alpha\beta}t^{\gamma\delta}_b$.

The difference tensor K can be fundamentally related to the bitorsion of the given connection one-form ω on L^2M. The local components of the pullback of the bitorsion structure equation in a local section σ of L^2M (see (2.9)] can be recast in a coordinated gauge of LM as

$$K^{a}_{[\mu\alpha\beta]} = \Theta^{a}_{[\mu\alpha\beta]} \tag{4.2}$$

The above follows from equation (4.1) in conjunction with definition (3.2), and it is valid in any coordinated section of LM. This fundamental relation between the bitorsion of a given connection one-form on L^2M and the difference form on L^2M will be important in the later formalism.

Any difference in connections leads to a corresponding relation between the respective curvature tensors [see, for example, Schouten (1954)]. Of particular interest is the associated form of (4.1), namely $\Gamma^{\gamma\delta}_{\mu\alpha\beta} = \mathring{\Gamma}^{\gamma\delta}_{\mu\alpha\beta} + K^{\gamma\delta}_{\mu\alpha\beta}$, which leads immediately to

$$R^{\theta\phi}_{\mu\nu\gamma\delta} = \mathring{R}^{\theta\phi}_{\mu\nu\gamma\delta} + P^{\theta\phi}_{\mu\nu\gamma\delta} \tag{4.3}$$

Here, $\mathring{R}$ are the local components of the biframe curvature which are induced from the Riemannian curvature on LM [see (1.6)], while R are the local components of a general biframe curvature associated with the given connection ω on L^2M. The tensor P is a new difference tensor which is constructed from K and its covariant derivatives. It takes the explicit form

$$P^{\theta\phi}_{\mu\nu\gamma\delta} = \nabla_\mu K^{\theta\phi}_{\nu\gamma\delta} - \nabla_\nu K^{\theta\phi}_{\mu\gamma\delta} + K^{\theta\phi}_{\mu\sigma\rho} K^{\sigma\rho}_{\nu\gamma\delta} - K^{\theta\phi}_{\nu\sigma\rho} K^{\sigma\rho}_{\mu\gamma\delta} \tag{4.4}$$

The Ricci tensor of a general biframe curvature R as given in (4.3) takes the form

$$R^{\alpha\beta} \stackrel{d}{=} \mathring{g}^{\alpha\nu\gamma\delta} R^{\mu\beta}_{\mu\nu\gamma\delta} = \mathring{R}^{\alpha\beta} + P^{\alpha\beta} \tag{4.5}$$

where we have used relation (3.3). For later applications we define

$$G^{\alpha\beta} \stackrel{d}{=} R^{(\alpha\beta)} \qquad \text{and} \qquad S^{\alpha\beta} \stackrel{d}{=} -P^{(\alpha\beta)}$$

so that the symmetrized part of (4.5) takes the form

$$G^{\alpha\beta} = \tilde{G}^{\alpha\beta} - S^{\alpha\beta} \tag{4.6}$$

where we have again used relation (3.3). Here $\tilde{G}^{\alpha\beta}$ are the local components of the Einstein tensor associated with the given Riemannian connection $\tilde{\omega}$ on LM.

Equation (4.6) gives a precise relation between contractions of a general biframe curvature and the Einstein tensor of a Riemannian spacetime. The tensor $S^{\alpha\beta}$ can, in some cases, act as a "geometrical source" for the Einstein tensor. As a trivial demonstration of this idea, we note that if $G^{\alpha\beta} = 0$, then (4.6) reduces to $\tilde{G}^{\alpha\beta} = S^{\alpha\beta}$. Thus, biframe vacuum spacetimes correspond to the Riemannian (frame bundle) Einstein equations with geometrical sources. This type of relation is not new. For example, the decomposition of the generalized Einstein tensor described above is analogous to that which occurs in typical U_4 theories of gravitation (that is, theories based on a metric connection on LM with nonzero linear torsion) [see, for example, Hehl *et al.* (1976)]. However, what is new in the present formalism is that the particular decomposition as given in (4.6) uniquely relates the Riemannian vector geometry associated with LM with the bivector geometry associated with L^2M.

In the following sections we will present a geometrization of the coupled Einstein–Maxwell equations with sources within the arena of the bundle of biframes. The new geometrical objects which allow this geometrization are the tensor $S^{\alpha\beta}$, as given in (4.6), and the local components of the bitorsion $\Theta^{\alpha}_{\mu\alpha\beta}$, as given in (4.2). Both of these objects are built from the difference form K, the metric g, and their derivatives. Furthermore, the fundamental decomposition of the symmetric part of the biframe Ricci tensor which occurs in (4.6) will be shown to lead to a geometrization of the Einstein equation with sources in Section 6.

We have shown that the assumptions of a Riemannian connection one-form on LM and a general connection one-form on L^2M necessarily imply the existence, at least locally, of an associated difference form K. *Throughout the rest of this paper we will denote a four-dimensional spacetime manifold M with a Riemannian connection one-form on LM and a general connection one-form on L^2M by a triple (M, g, K).*

5. THE STANDARD RMW THEORY

We next recall some basic facts from RMW theory (Rainich, 1925; Misner and Wheeler, 1957). A source free Einstein–Maxwell spacetime is any four-dimensional Riemannian spacetime which satisfies

$$\tilde{G}_{\mu\nu} = f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \ \nu} \tag{5.1}$$

$$\nabla_{\alpha} f^{*\lambda\alpha} = 0 \tag{5.2}$$

$$\nabla_{\alpha} f^{\lambda\alpha} = 0 \tag{5.3}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor and $f_{\mu\nu}$ is the Maxwell field strength. Furthermore, a nonnull field satisfies $f^{*}_{\theta\phi} f^{\theta\phi} \neq 0$.

The RMW theory provides a method of geometrizing such spacetimes. The following conditions are both necessary and sufficient for an arbitrary four-dimensional Riemannian spacetime (M, g) to be equivalent to a nonnull, source-free Einstein–Maxwell spacetime:

$$\tilde{G} = 0 \tag{5.4}$$

$$\tilde{G}^{\beta}_{\alpha}\tilde{G}^{\gamma}_{\beta}=\tfrac{1}{4}\delta^{\gamma}_{\alpha}\tilde{G}_{\theta\phi}\tilde{G}^{\theta\phi} \tag{5.5}$$

$$\tilde{G}_{00}\geq 0 \tag{5.6}$$

$$\alpha_{\mu}=\partial_{\mu}\alpha \tag{5.7}$$

where

$$\alpha_{\mu}=\mathscr{E}_{\mu\lambda\gamma\nu}\left\{\frac{\tilde{G}^{\nu}_{\sigma}\nabla^{\gamma}\tilde{G}^{\lambda\sigma}}{\tilde{G}_{\theta\phi}\tilde{G}^{\theta\phi}}\right\} \tag{5.8}$$

In the above relations $\tilde{G}_{\theta\phi}\tilde{G}^{\theta\phi}\neq 0$.

Nonnull Einstein–Maxwell spacetimes are geometrized in that conditions (5.4)–(5.8) are purely geometrical relations stated completely in terms of g and its derivatives. In particular, the nongeometrical Maxwell field strength $f_{\mu\nu}$ does not explicitly appear in these conditions. This nonappearance of $f_{\mu\nu}$ in equations (5.4)–(5.8) is a strength of the RMW theory in that the conditions can be stated completely in terms of the metric. On the other hand, the physical Maxwell field $f_{\mu\nu}$ does *not* play a fundamental geometrical role in the theory.

Any four-dimensional Riemannian geometry which satisfies (5.4)–(5.6) with $\tilde{G}_{\theta\phi}\tilde{G}^{\theta\phi}\neq 0$ will be called a nonnull *algebraic RMW spacetime* (ARMW), while (5.7)–(5.8) will be referred to as the RMW *differential condition.* The vector α_{μ} in (5.8) is called the *complexion vector.*

The electromagnetic field strength can be recovered in the RMW theory. Given an algebraic RMW spacetime, there exist naturally defined geometrical bivectors $\xi_{\alpha\beta}$ and its dual $\xi^{*}_{\alpha\beta}$. The bivector $\xi_{\alpha\beta}$ is the so-called *extremal Maxwell square root* of the Einstein tensor (Rainich, 1925; Misner and Wheeler, 1957).

For ARMW spacetimes, new bivectors $\Sigma_{\alpha\beta}$ and $\Sigma^{*}_{\alpha\beta}$ can be constructed by a *duality rotation*, that is,

$$\Sigma_{\alpha\beta}=\xi_{\alpha\beta}\cos\alpha+\xi^{*}_{\alpha\beta}\sin\alpha \tag{5.9}$$

$$\Sigma^{*}_{\alpha\beta}=\xi^{*}_{\alpha\beta}\cos\alpha-\xi_{\alpha\beta}\sin\alpha \tag{5.10}$$

The algebraic conditions (5.4)–(5.6) are necessary and sufficient to guarantee that the new bivectors $\Sigma_{\alpha\beta}$ and $\Sigma^{*}_{\alpha\beta}$ for each complexion angle α satisfy the quadratic form

$$\begin{aligned}\tilde{G}_{\mu\nu}&=\xi_{\alpha\mu}\xi^{\alpha}_{\nu}+\xi^{*}_{\alpha\mu}\xi^{*\alpha}_{\nu}\\&=\Sigma_{\alpha\mu}\Sigma^{\alpha}_{\nu}+\Sigma^{*}_{\alpha\mu}\Sigma^{*\alpha}_{\nu}\end{aligned} \tag{5.11}$$

as in (5.1).

Thus, ARMW spacetimes guarantee the quadratic structure (5.1), but the Maxwell equations need not be satisfied. The extra condition necessary and sufficient for the source-free Maxwell equations to be satisfied is precisely the RMW differential condition (5.7)-(5.8).

Recall that when the differential condition (5.7)-(5.8) is satisfied, that is, when $\alpha_\mu = \partial_\mu \alpha$, the Maxwell equations (5.2) and (5.3) can be written in terms of the extremal Maxwell square root as

$$\nabla_\mu \xi^{*\lambda\mu} - (\partial_\mu \alpha)\xi^{\lambda\mu} = 0 \tag{5.12}$$

$$\nabla_\mu \xi^{\lambda\mu} + (\partial_\mu \alpha)\xi^{*\lambda\mu} = 0 \tag{5.13}$$

The relations (5.12) and (5.13) will be called the *RMW extremal field equations.* Thus, (5.12) and (5.13) are equivalent to the Maxwell equations in that a duality rotation on the bivectors occurring in (5.12) and (5.13) will produce the source-free Maxwell equations (5.2) and (5.3).

6. THE EINSTEIN EQUATION AS AN ASPECT OF BIFRAME BUNDLE GEOMETRY

To extend the original RMW program to include Einstein–Maxwell spacetimes with sources, we extend the geometrical arena from four-dimensional Riemannian spacetimes (M, g) to triples (M, g, K). In this section we provide a geometrization of the Einstein equation associated with Einstein–Maxwell spacetimes with sources. A key step in this geometrization is to extend the notion of ARMW spacetimes from the usual linear frame bundle LM to the bundle of biframes L^2M. As motivation for this extension, consider the following discussion.

The Einstein equation associated with an Einstein–Maxwell spacetime with sources [see (1.1)] can be written as

$$\tilde{G}_{\mu\nu} = T^{\text{total}}_{\mu\nu} = (f_{\alpha\mu} f^{\alpha}_{\nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\nu}) + T^{s}_{\mu\nu} \tag{6.1}$$

Here $T^s_{\mu\nu}$ represents the nongeometrical, nonelectromagnetic source stress tensor. For example, this part of the stress tensor for a charged fluid takes the form $T^s_{\alpha\beta} = \mu u_\alpha u_\beta$, where u_α is the four-velocity of the fluid and μ is the energy density [see, for example, Synge (1960)].

Note that the difference $(\tilde{G}_{\mu\nu} - T^s_{\mu\nu})$ has the quadratic bivector structure which is characteristic of source-free Einstein–Maxwell spacetimes. That is, given any Einstein–Maxwell spacetime with sources, we can write $(\tilde{G}_{\mu\nu} - T^s_{\mu\nu}) = (f_{\alpha\mu} f^{\alpha}_{\nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\nu})$. To extend the notion of ARMW spacetimes to the bundle of biframes, we first note that each (M, g, K) necessarily implies the decomposition [see (4.6)]

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} - S_{\mu\nu} \tag{6.2}$$

which has the same general form as described above.

A *biframe algebraic Rainich spacetime* (BAR) is any triple (M, g, K) such that the symmetrized portion of the biframe Ricci tensor $G_{\mu\nu}$ given in (6.2) satisfies the following conditions [see (5.4)-(5.6)]:

$$G = 0 \tag{6.3}$$

$$G_\alpha^\beta G_\beta^\gamma = \tfrac{1}{4}\delta_\alpha^\gamma G_{\theta\phi} G^{\theta\phi} \tag{6.4}$$

$$G_{00} \geq 0 \tag{6.5}$$

Furthermore, the spacetime will be referred to as nonnull for $G_{\theta\phi}G^{\theta\phi} \neq 0$, and null for $G_{\theta\phi}G^{\theta\phi} = 0$. As in the standard RMW formalism, the algebraic conditions (6.3)-(6.5) are necessary and sufficient to guarantee that the tensor $G_{\mu\nu}$ can be written in the form

$$\begin{aligned} G_{\mu\nu} &= \xi_{\alpha\mu}\xi^\alpha_{\ \nu} + \xi^*_{\alpha\mu}\xi^{*\alpha}_{\ \nu} \\ &= f_{\alpha\mu} f^\alpha_{\ \nu} + f^*_{\alpha\mu} f^{*\alpha}_{\ \nu} \end{aligned} \tag{6.6}$$

Here, $\xi_{\mu\nu}$ is now the *extremal Maxwell square root* of $G_{\mu\nu}$, with dual $\xi^*_{\mu\nu}$, and the cobivectors $f_{\mu\nu}$ and $f^*_{\mu\nu}$ are obtained by a duality rotation [see (5.9) and (5.10)] of the extremal fields through a complexion angle α.

Note that if a triple (M, g, K) is a BAR spacetime, then it follows from the fundamental decomposition of the biframe Ricci tensor as given in (6.2) that the Einstein tensor associated with the four-dimensional Riemannian geometry necessarily satisfies

$$\tilde{G}_{\mu\nu} = (f_{\alpha\mu} f^\alpha_{\ \nu} + f^*_{\alpha\mu} f^{*\alpha}_{\ \nu}) + S_{\mu\nu} \tag{6.7}$$

Clearly, this is the same form as that given in relation (6.1), where the purely geometrical tensor $S_{\mu\nu}$ now plays the role previously played by the nongeometrical, nonelectromagnetic source stress tensor $T^s_{\mu\nu}$. We refer to relation (6.7) as the *Einstein equation with geometrical sources.* The algebraic conditions (6.3)-(6.5) on an arbitrary triple (M, g, K) are both necessary and sufficient to guarantee the form of the Riemannian Einstein tensor as given in (6.7).

7. THE BIFRAME BUNDLE AS A NATURAL GEOMETRICAL ARENA FOR THE RMW THEORY

The biframe bundle is a natural geometrical arena in which to extend the RMW program. The intuitive idea of the construction described below is as follows. If one is given a BAR spacetime, then, as discussed in the previous section, there exist naturally defined geometrical bivectors, namely the extremal Maxwell square root $\xi_{\alpha\beta}$ of the biframe Ricci tensor and its dual $\xi^*_{\alpha\beta}$. As will be shown below, these bivectors can be used to define special sections of L^2M. Furthermore, it will be shown that duality rotations correspond precisely to special section changes on the biframe bundle.

Thus, to model the RMW problem on the biframe bundle, consider the set of all BAR spacetimes. Each such BAR spacetime leads to the geometrical bivectors $\xi_{\alpha\beta}$ and $\xi^*_{\alpha\beta}$. An equivalence class $[{}^*\xi]$ of sections of L^2M^* can then be defined as follows. Two sections ${}^*\xi$, ${}^*\eta \in [{}^*\xi]$ are equivalent if

$$ {}^*\xi(p) = (p, (\xi_{\alpha\beta}, -\xi^*_{\alpha\beta}, \tau^A)_p), \qquad {}^*\eta(p) = (p, (\xi_{\alpha\beta}, -\xi^*_{\alpha\beta}, \bar{\tau}^A)_p) $$

$$ \forall p \in U, \qquad U \subseteq M $$

The (τ^A) $(A = 3, 4, 5, 6)$ are, for our purposes, arbitrary cobivectors picked to complete the cobiframe. That is, in this construction we will only be concerned with the one-two blades, namely, in the first and second cobivectors of the cobiframe. Any ${}^*\xi \in [{}^*\xi]$ will be called an *extremal gauge* of L^2M^*. The corresponding dual gauge of L^2M will be labeled ξ.

Consistent with the above construction, we next define a special gauge transformation. Let $h\colon U \to Gl(6)$, $U \subseteq M$, be defined by

$$ h(p) = \begin{pmatrix} \cos\alpha(p) & \sin\alpha(p) & 0 \\ -\sin\alpha(p) & \cos\alpha(p) & 0 \\ 0 & 0 & I_4 \end{pmatrix} \tag{7.1} $$

where $\alpha\colon U \to \mathbb{R}$ and the above holds for all $p \in U$. The 4×4 identity matrix I_4 could be replaced by a general $Gl(4)$ matrix. However, again we are interested here only in the first and second blades and thus we simplify.

Given ${}^*\xi$ an extremal section of L^2M^*, a new section ${}^*\Sigma = {}^*\xi \cdot h$ at a point $p \in M$ has the form ${}^*\Sigma(p) = (p, (\Sigma_{\alpha\beta}, -\Sigma^*_{\alpha\beta}, \tau^A)_p)$. Here, $\Sigma_{\alpha\beta}$ and $\Sigma^*_{\alpha\beta}$ are precisely a duality rotation of the extremal fields $\xi_{\alpha\beta}$ and $\xi^*_{\alpha\beta}$ as in (5.9) and (5.10). *Thus, duality rotations correspond to special section changes of L^2M^*.*

This model, in which the extremal fields are part of a bivector basis and duality rotations are special $Gl(6)$ transformations on this basis, helps to clarify several aspects related to the RMW problem. For example, in typical discussions concerning the RMW problem it is usually shown that expressions such as $\xi_{\alpha\beta}\xi^{\alpha\beta}$ and $\xi^*_{\alpha\beta}\xi^{\alpha\beta}$ are not duality invariant. From the above discussion, ${}^*\xi(p) = (p, (\xi_{\alpha\beta}, -\xi^*_{\alpha\beta}, \tau^A)_p)$ and thus the expressions in question are

$$ \xi_{\alpha\beta}\xi^{\alpha\beta} = \tau^{(1)}_{\alpha\beta}\tau^{\alpha\beta(1)} \tag{7.2} $$

$$ -\xi^*_{\alpha\beta}\xi^{\alpha\beta} = \tau^{(2)}_{\alpha\beta}\tau^{\alpha\beta(1)} \tag{7.3} $$

These correspond to components of $\tau^a_{\alpha\beta}\tau^{\alpha\beta b}$, and clearly are not invariant under $Gl(6)$ transformations (Norris and Davis, 1979).

8. THE MAXWELL EQUATIONS AS AN ASPECT OF BIFRAME BUNDLE GEOMETRY

Geometrical sources for the Maxwell equations arise in an extremely natural manner in terms of biframe bundle geometry. Each BAR spacetime gives rise to an equivalence class $[{}^*\xi]$ of extremal sections of L^2M^*. If ${}^*\xi$ is any extremal section of L^2M^* associated with a given BAR spacetime, then the first and second components of the pullback of the bitorsion structure equation in the gauge ${}^*\xi$ [see (2.9)] can be recast in a coordinated section of LM in the form [for further details see Hammon and Norris (1990*c*)]

$$\nabla_\mu \xi^{*\lambda\mu} - {}_\xi\alpha_\mu \xi^{\lambda\mu} + {}_\xi\beta_\mu \xi^{*\lambda\mu} = {}^*_\xi\Theta^{\lambda(1)} \tag{8.1}$$

$$\nabla_\mu \xi^{\lambda\mu} + {}_\xi\alpha_\mu \xi^{*\lambda\mu} + {}_\xi\beta_\mu \xi^{\lambda\mu} = {}^*_\xi\Theta^{\lambda(2)} \tag{8.2}$$

These relations are referred to as the *generalized extremal identities*, while α_μ and β_μ are the *generalized complexion vectors.*

Clearly, the generalized extremal identities are a generalization in form of the standard RMW extremal field equations which appear in (5.12)–(5.13). Note that the bitorsion ${}^*_\xi\Theta^{\lambda a}$ plays the role of a geometrical source in these identities. If the given BAR spacetime is a nonnull spacetime, the generalized extremal identities can be solved for the generalized complexion vectors. Indeed, using special bivector identities [see, for example, Misner and Wheeler (1957)], we obtain

$${}_\xi\alpha_\mu = \mathscr{E}_{\mu\lambda\alpha\nu}\left\{\frac{G^\nu_\sigma \nabla^\alpha G^{\lambda\sigma}}{G_{\theta\phi}G^{\theta\phi}}\right\} + \frac{2({}^*_\xi\Theta^{\lambda(1)}\xi_{\lambda\mu} + {}^*_\xi\Theta^{\lambda(2)}\xi^*_{\lambda\mu})}{G_{\theta\phi}G^{\theta\phi}} \tag{8.3}$$

$${}_\xi\beta_\mu = 4\left\{\frac{G^\sigma_\mu \nabla_\alpha G^\alpha_\sigma}{G_{\theta\phi}G^{\theta\phi}}\right\} + \frac{2({}^*_\xi\Theta^{\lambda(1)}\xi^*_{\lambda\mu} - {}^*_\xi\Theta^{\lambda(2)}\xi_{\lambda\mu})}{G_{\theta\phi}G^{\theta\phi}} \tag{8.4}$$

The above relations are a generalization in form of the single standard RMW complexion vector given in (5.8). Note that the generalized complexion vector ${}_\xi\beta_\mu$ given in (8.4) does not appear in the standard RMW formalism. The new terms which appear in both (8.3) and (8.4) involving the bitorsion are due to the geometrical currents which occur in (8.1) and (8.2). Lastly, we emphasize that the tensor $G_{\mu\nu}$ which appears in (8.3) and (8.4) is the biframe Ricci tensor rather than the Riemannian Einstein tensor $\tilde{G}_{\mu\nu}$ which occurs in relation (5.8). The reduction of the generalized complexion vectors back to the form of the standard RMW complexion vector will be discussed in the following section.

Clearly, the generalized extremal identities which appear in (8.1) and (8.2) are more general than the Maxwell equations. However, suitable restrictions can be made on these identities such that they reduce to the

Maxwell equations with bitorsion sources. These special restrictions are precisely a generalization of the RMW differential condition and they occur in the following theorem.

Theorem 8.1. Assume a Riemannian linear connection on LM, a general linear connection on L^2M, and assume that the corresponding triple (M, g, K) is also a BAR spacetime. Let $^*\xi$ be an extremal gauge of L^2M^* corresponding to the given BAR spacetime and let ξ be a corresponding dual gauge of L^2M. If $f = \xi \cdot h$ is a new section of L^2M [where h is as in (7.1)], then in any coordinated section of LM

$$ {}_\xi\alpha_\mu = \partial_\mu \alpha \qquad \text{and} \qquad {}_\xi\beta_\mu = 0 \tag{8.5}$$

if and only if

$$\nabla_\alpha f^{*\lambda\alpha} = {}^*_f\Theta^{\lambda(1)} \tag{8.6}$$

$$\nabla_\alpha f^{\lambda\alpha} = {}^*_f\Theta^{\lambda(2)} \tag{8.7}$$

Furthermore, in this case, the first and second components of the pullback of the bitorsion Bianchi identity in the gauge f of L^2M [see (2.10)] are equivalent to

$$\nabla_\alpha ({}^*_f\Theta)^{\alpha(1)} \equiv 0 \tag{8.8}$$

$$\nabla_\alpha ({}^*_f\Theta)^{\alpha(2)} \equiv 0 \tag{8.9}$$

A slightly different version of the above theorem appears in Hammon and Norris (1990*c*), and the reader should consult that reference for details of the proof.

The special case in which ${}_\xi\alpha_\mu = \partial_\mu \alpha$ and ${}_\xi\beta_\mu = 0$ guarantees the form of the equations as given in (8.6) and (8.7), and we will refer to these relations as the *Maxwell equations with geometrical sources.* These geometrical equations follow from the bitorsion structure equation on the biframe bundle. Furthermore, the bitorsion Bianchi identity, under these same restrictions, guarantees conservation of the sources. This result is an analog of the well-known source conservation which occurs in standard general relativity.

The results of Theorem 8.1 are clearly more general than the standard RMW theory in that the standard RMW problem deals only with the source-free Maxwell equations. The algebraic structure associated with BAR spacetimes in conjunction with the geometrical richness of the biframe bundle has led to Maxwell equations with conserved geometrical sources.

9. AN EXTENSION OF THE RMW PROGRAM TO INCLUDE SOURCES

The original RMW theory geometrized source-free Einstein–Maxwell spacetimes in terms of the vector geometry of the linear frame bundle *LM*. Indeed, the original program provided the necessary and sufficient conditions on an arbitrary four-dimensional Riemannian spacetime (M, g) in order for this spacetime to be a nonnull, source-free Einstein–Maxwell spacetime.

In this paper we have shown that the bundle of biframes is a natural geometrical arena in which to reformulate and extend the standard RMW program. The assumptions of a Riemannian connection one-form on *LM* and a general connection one-form on L^2M necessarily imply a triple (M, g, K), where K is a natural difference form described in Section 4. The following is an extension of the original RMW theorem to the bundle of biframes and it provides the necessary and sufficient conditions on a arbitrary triple (M, g, K) in order for this triple to be a nonnull Einstein–Maxwell spacetime with geometrical sources.

Theorem 9.1. Assume a Riemannian linear connection on *LM*, a general linear connection on L^2M, and assume that the corresponding triple (M, g, K) is also a nonnull BAR spacetime such that it satisfies relations (6.3)–(6.5). Let $\xi \in [\xi]$ be any extremal gauge of L^2M associated with the given BAR spacetime. If $f = \xi \cdot h$ is a new section of L^2M, [where h is as in (7.1)], then in a coordinated section of *LM*

$$
{}_{\xi}\alpha_{\mu} = \partial_{\mu}\alpha \qquad \text{and} \qquad {}_{\xi}\beta_{\mu} = 0 \tag{9.1}
$$

if and only if

$$
\tilde{G}_{\mu\nu} = (f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \nu}) + S_{\mu\nu} \tag{9.2}
$$

$$
\nabla_{\mu} f^{*\lambda\mu} = {}^{*}_{f}\Theta^{\lambda(1)} \tag{9.3}
$$

$$
\nabla_{\mu} f^{\lambda\mu} = {}^{*}_{f}\Theta^{\lambda(2)} \tag{9.4}
$$

Here

$$
{}_{\xi}\alpha_{\mu} = \mathscr{E}_{\mu\lambda\alpha\nu}\left\{\frac{G^{\nu}_{\sigma}\nabla^{\alpha}G^{\lambda\sigma}}{G_{\theta\phi}G^{\theta\phi}}\right\} + \frac{2({}^{*}_{\xi}\Theta^{\lambda(1)}\xi_{\lambda\mu} + {}^{*}_{\xi}\Theta^{\lambda(2)}\xi^{*}_{\lambda\mu})}{G_{\theta\phi}G^{\theta\phi}} \tag{9.5}
$$

$$
{}_{\xi}\beta_{\mu} = 4\left\{\frac{G^{\sigma}_{\mu}\nabla_{\alpha}G^{\alpha}_{\sigma}}{G_{\theta\phi}G^{\theta\phi}}\right\} + \frac{2({}^{*}_{\xi}\Theta^{\lambda(1)}\xi^{*}_{\lambda\mu} - {}^{*}_{\xi}\Theta^{\lambda(2)}\xi_{\lambda\mu})}{G_{\theta\phi}G^{\theta\phi}} \tag{9.6}
$$

Furthermore, in this case the first and second components of the pullback of the bitorsion Bianchi identity in the section f of L^2M take the form

$$\nabla_\mu ({}^*_f\Theta)^{\mu(1)} \equiv 0 \tag{9.7}$$

$$\nabla_\mu ({}^*_f\Theta)^{\mu(2)} \equiv 0 \tag{9.8}$$

Proof. The main results follow from Theorem 8.1 in conjunction with equations (6.3)-(6.6), as well as relations (6.7), (8.3), and (8.4). ■

Any triple (M, g, K) which satisfies relations (9.2)-(9.4) for some section $f = \xi \cdot h$ (a duality rotated section) of L^2M will be called an *Einstein-Maxwell spacetime with geometrical sources.* Clearly, the bitorsion ${}^*_f\Theta^{\lambda a}$ acts as a geometrical source for the Maxwell equations, while the tensor $S_{\mu\nu}$ acts as a nonelectromagnetic, geometrical source for the Riemannian Einstein equation. As discussed in Section 4, the tensors ${}^*_f\Theta^{\lambda a}$ and $S_{\mu\nu}$ are constructed from the metric g, the difference form K, and their derivatives.

The results of Theorem 9.1 are clearly more general than the standard RMW theorem, and we discuss the reduction of this theorem back to the standard RMW formalism below. We first note that the formalism introduces a second complexion type of vector ${}_\xi\beta_\mu$ as given in (9.6), as well as a generalization of the standard RMW complexion vector ${}_\xi\alpha_\mu$ which appears in (9.5). The new terms involving the bitorsion which appear in (9.5) and (9.6) are due to the presence of geometrical electric and magnetic currents which occur in the Maxwell equations.

A reduction of the general results of Theorem 9.1 back to the standard RMW formalism can occur as follows. If the geometrical difference form K vanishes, then both $S_{\mu\nu}$ and ${}^*\Theta^{\lambda a}$ (see Section 4) also vanish and it follows from relation (4.6) that $G_{\mu\nu} = \tilde{G}_{\mu\nu}$. As a consequence of these restrictions, the Einstein-Maxwell equations with geometrical sources given in (9.2)-(9.4) reduce precisely to the standard source-free Einstein-Maxwell equations, which are, again, those equations encompassed by the usual RMW formalism.

Simultaneously, when the difference form K vanishes, the relations of Theorem 9.1 reduce to those of the standard RMW theorem. In particular, the conditions on $G_{\mu\nu}$ for a BAR spacetime, namely (6.3)-(6.5), are now restrictions on $\tilde{G}_{\mu\nu}$ and, moreover, they are the same relations which occur in (5.4)-(5.6) for an ARMW spacetime. Next, for this special case, the generalized complexion vector ${}_\xi\alpha_\mu$ which occurs in (9.5) reduces in form to the standard RMW complexion vector given in (5.8), and the differential condition given in (9.1) is therefore the same as that given in (5.7). Note that in this special case the new complexion vector ${}_\xi\beta_\mu$ which occurs in (9.6) vanishes identically via the doubly contracted Bianchi identities of the Riemannian curvature tensor on LM. That is, in this special case the new "differential condition," namely that ${}_\xi\beta_\mu = 0$ [see (9.1)], is identically satisfied and imposes no new restrictions on the Riemannian spacetime. This is why the new complexion vector ${}_{\xi}\beta_{\mu}$ never appeared in the standard RMW formalism.

Hence, when the difference form K vanishes, the Einstein–Maxwell equations with geometrical sources reduce to the standard source-free Einstein–Maxwell equations and, simultaneously, the formalism of Theorem 9.1 *reduces precisely to the usual RMW theorem. The vanishing of the difference form K also has geometrical significance and we will comment on this in the following section.*

A less restrictive reduction, rather than the difference form K vanishing, is also possible. First, note that if the bitorsion ${}^{*}\Theta^{\lambda a}$ vanishes, then $S_{\mu\nu}$ need not be zero. This corresponds physically to zero electromagnetic sources in the Maxwell equations [that is, (9.3)–(9.4)] while having nonzero, nonelectromagnetic sources for the Einstein tensor [see (9.2)]. This special case of the coupled Einstein–Maxwell equations with geometrical sources is certainly physically reasonable.

Another intermediate case of interest is less straightforward. The tensor $S_{\mu\nu}$ can vanish, while the bitorsion ${}^{*}\Theta^{\lambda a}$ need not be zero. This would seem to imply a potential flaw in the theory, as it would allow nonzero electromagnetic currents for the coupled Einstein–Maxwell equations which did not also appear as gravitational sources in the Einstein equation. However, it can be shown that $S_{\mu\nu}$ being zero in conjunction with the conditions that ${}_{\xi}\alpha_{\mu} = \partial_{\mu}\alpha$ and ${}_{\xi}\beta_{\mu} = 0$ in fact forces the first and second components of the bitorsion to vanish, that is, ${}^{*}_{f}\Theta^{\lambda(1)} = 0$ and ${}^{*}_{f}\Theta^{\lambda(2)} = 0$. Hence, the generalized RMW formalism obeys the empirical relation that electromagnetic currents for the coupled Einstein–Maxwell equations with geometrical sources can vanish while the nonelectromagnetic gravitational sources do not, but not vice versa.

10. CONCLUSIONS

The original RMW program provides a geometrization of source-free Einstein–Maxwell spacetimes within the arena of standard four-dimensional Riemannian geometry. The objective of this paper has been to show that the original RMW theory can be extended to include Einstein–Maxwell spacetimes with geometrical sources by generalizing the geometrical arena to a new principal fiber bundle, namely the bundle of biframes L^2M. A detailed construction of the biframe bundle, including developments of its associated differential geometry, can be found in Hammon and Norris (1990*b*). A review of the main results concerning this geometry was presented in Section 2 of this paper.

A key geometrical development which allows the above geometrization is the introduction of a difference form K on L^2M. Indeed, we showed in Section 4 that the assumptions of a Riemannian connection one-form on LM and a general connection one-form on L^2M necessarily imply the existence, at least locally, of a difference form K. Any four-dimensional spacetime manifold M, with the previous geometrical assumptions on the principal fiber bundles LM and L^2M over M, is denoted as a triple (M, g, K), where g is the metric tensor.

The Einstein equation associated with the standard coupled Einstein–Maxwell equations with sources [see (1.1)] was geometrized in a natural manner within this new geometrical arena. Indeed, we showed in Section 4 that any given triple (M, g, K) necessarily implies the fundamental decomposition of the biframe Ricci tensor $G_{\mu\nu}$ as given in (4.6), namely

$$G_{\mu\nu} = \tilde{G}_{\mu\nu} - S_{\mu\nu} \tag{10.1}$$

where $\tilde{G}_{\mu\nu}$ is the Riemannian Einstein tensor and $S_{\mu\nu}$ is a new difference tensor constructed from the metric g, the difference form K, and their derivatives.

The next step taken in this process of geometrization was to extend the notion of ARMW spacetimes from the linear frame bundle LM to the biframe bundle L^2M. This was accomplished in Section 6 by placing the algebraic conditions (6.3)–(6.5) on the biframe Ricci tensor $G_{\mu\nu}$, rather than the standard relations (5.4)–(5.6) on the Einstein tensor $\tilde{G}_{\mu\nu}$. Any triple (M, g, K) which satisfies the algebraic conditions (6.3)–(6.5) is called a biframe algebraic Rainich spacetime (BAR). The notion of BAR spacetimes in conjunction with the above natural decomposition of the biframe Ricci tensor guarantees that the Riemannian Einstein tensor takes the form [see (6.7)]

$$\tilde{G}_{\mu\nu} = (f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \nu}) + S_{\mu\nu} \tag{10.2}$$

where $f_{\mu\nu}$ is now a Maxwell square root of the biframe Ricci tensor $G_{\mu\nu}$. Clearly, this relation is the same form as the standard field equation which occurs in (1.1), except all sources are geometrical. The introduction of the triples (M, g, K) and the corresponding decomposition of the Einstein tensor as in (10.2) provides a purely geometrical extension of the earlier works of Hammon and Norris (1990*a,c*).

The biframe bundle is a natural geometrical arena in which to reformulate and extend the original RMW theory. Indeed, we showed in Section 7 that each BAR spacetime guarantees an entire equivalence class of extremal sections of L^2M^*. Furthermore, the concept of duality rotations corresponds precisely to special changes of these sections.

The Maxwell equations associated with the standard coupled Einstein-Maxwell equations with sources, namely (1.2) and (1.3), are geometrized in a natural manner via the bitorsion three-form associated with a given connection one-form on the bundle of biframes L^2M. Indeed, we showed in Section 8 that the pullback of the bitorsion structure equation from LM to M in any extremal section associated with a given BAR spacetime takes the form of differential bivector identities on M [see (8.1)-(8.2)]. These generalized extremal identities are clearly more general than the standard RMW extremal field equations which occur in relations (5.12)-(5.13).

We showed in Theorem 8.1 that suitable restrictions can be made on the generalized extremal identities such that they take the form [see (8.6)-(8.7)]

$$\nabla_\alpha f^{*\lambda\alpha} = {}^{*}_{f}\Theta^{\lambda(1)} \tag{10.3}$$

$$\nabla_\alpha f^{\lambda\alpha} = {}^{*}_{f}\Theta^{\lambda(2)} \tag{10.4}$$

where ${}^{*}_{f}\Theta^{\lambda a}$ are the local components of the bitorsion. Furthermore, we showed that the above special restrictions are a generalization of the standard RMW differential condition. Clearly, relations (10.3)-(10.4) are the same form as the standard field equations which occur in (1.2)-(1.3), except all currents are geometrized via the bitorsion. Moreover, we also showed that these geometrical sources are conserved as a consequence of the bitorsion Bianchi identity.

Any triple (M, g, K) which satisfies relations (10.2)-(10.4) is called an Einstein-Maxwell spacetime with geometrical sources. The new geometrical objects which play the role of sources for these coupled equations are the bitorsion ${}^*\Theta^{\lambda a}$ and the curvature difference tensor $S_{\mu\nu}$. Specifically, components of the bitorsion play the role of geometrical currents for the Maxwell equations, while $S_{\mu\nu}$ represents a geometrization of all nonelectromagnetic sources for the Einstein equation. Both ${}^*\Theta^{\lambda a}$ and $S_{\mu\nu}$ are constructed from the difference form K, the metric g, and their derivatives.

The introduction of the difference form K on L^2M thus allows a generalization of the original RMW theorem to include sources. Indeed, the theorem presented in Section 9 provides the necessary and sufficient conditions on an arbitrary triple (M, g, K) in order for this triple to be an Einstein-Maxwell spacetime with geometrical sources. As described above, the field equations associated with such spacetimes have the same form as those associated with the standard Einstein-Maxwell spacetimes, except that all sources are geometrized.

A reduction of the generalized RMW formalism back to the standard RMW theory can occur when the difference form K vanishes. This restriction has both physical and geometrical significance. The physical significance of the vanishing of the difference form K was discussed in Section 9. Indeed, we showed that when the difference form K is zero, the coupled Einstein-Maxwell equations with geometrical sources reduce precisely to the standard source-free equations associated with Riemannian spacetimes, and, simultaneously, the generalized RMW formalism, as given in Theorem 9.1, reduces term by term to the standard RMW theorem. The vanishing of the difference form K also has a geometrical interpretation. Indeed, when the difference form vanishes, the geometry on L^2M reduces to that which is only induced by the Riemannian geometry on LM. This special case helps to clarify the intimate relation between the traditional source-free case and standard Riemannian geometry.

A fundamental question which arises from the generalized RMW formalism is the following. Can a specific form for the general objects $^*\Theta^{\lambda a}$ and $S_{\mu\nu}$ be chosen, or derived, such that the general Einstein-Maxwell equations with geometrical sources reduce to a physically significant form? Preliminary results indicate that this is possible. Indeed, these results indicate that by adding new algebraic conditions on $S_{\mu\nu}$ and $^*\Theta^{\lambda a}$, in addition to those already on $G_{\mu\nu}$, it is possible to recover the coupled Einstein-Maxwell equations for a charged perfect fluid [see, for example, Lichnerowicz (1967)]. This type of procedure is certainly within the spirit of the original RMW program, as only geometrical algebraic field equations are imposed. We hope to comment on these preliminary results in more detail in a future publication.

A problem which is related to the above is the notion of exact solutions. In this paper we have demonstrated the necessary and sufficient conditions on an arbitrary triple (M, g, K) in order for it to be a nonnull Einstein-Maxwell spacetime with geometrical sources. In analogy to the original RMW program, these conditions are to be taken as field equations to solve not only for the metric g, but also for the difference form K. Presumably, each special form for the coupled Einstein-Maxwell equations, such as the charged perfect fluid alluded to above, will have its own specific form for K. Can an exact solution for these new field equations be found which will permit a determination of a specific form for the difference form K?

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